Stochastic Evaluation of Multi-State Coherent Systems (Mathematical Decision Making under Uncertainty and Ambiguity, and Related Topics)

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Abstract

In this paper we present some bounds for stochastic performances of multi-state systems, using associated probability measures on state spaces defined to be mathematically partially ordered sets. These bounds are obtained by using series decomposition of systems which is well known as max-min formula for the binary-state case.

Keywords: Multi-state coherent systems, Series systems, Ordered set, Lattice set, Stochastic Bounds

1. Introduction

A basic problem of reliability theory is to explain relationships among the reliability performances of systems and the components. Using Boolean functions, Mine [11] introduced the concept of monotone systems, in which all the state spaces of components and the systems were assumed to be $\{0,1\}$, so called binary state systems, where 0 and 1 denote the failure and the functioning states, respectively. "Monotone system" means that the more the number of functioning components is, the higher the performance level of the system is.

Mathematical aspects of binary state monotone systems have been fully explained by Birnbaum and Esary [3], Birnbaum, et al. [4] and Esary and Proschan [6]. Barlow and Proschan [1] have summarized the reliability studies of the binary state monotone systems. Pham [20] has compiled recent work about reliability engineering in which several formulae for solving practical reliability problems are given.

In many practical situations, however, systems and their components could take many other performance levels from the perfectly functioning state to the complete failure state, thus multi-state reliability models are needed for understanding and solving practical problems.

Multi-state systems were introduced in the context of cannibalization by Hirsch, Meisner and Boll [8] and Hochberg [9]. Mathematical studies of multi-state systems were first carried out by Barlow and Wu [2] and El-Neweihi, Proschan and Sethuraman [5]. Barlow and Wu [2] defined multi-state coherent systems, making use of the minimal path and cut sets of binary state systems. El-Neweihi, Proschan and Sethuraman [5] defined the multi-state systems assuming all the state spaces to be expressed commonly as $\{0,1,\cdots,M\}$. Their results were almost analogous to those of binary systems. Huang, Zuo and Fang [10] introduced the multi-state consecutive k-out-of-n systems and provided algorithms to evaluate the performance probabilities of the systems. Zuo, Hang and Kuo [26] defined a multi-state coherent systems assuming all the state spaces of the systems and components were the same finite totally ordered sets as El-Neweihi, Proschan and Sethuraman [5].

Ohi and Nishida [13], Ohi [18] have defined multi-state systems, assuming that the state spaces of components and systems are totally ordered finite sets which have not necessarily the same cardinal numbers, where they have fully examined order and stochastic properties of multi-state systems and almost all the concepts and results of binary-state cases are generalized to totally ordered cases.

Yu, Koren and Guo [25] have presented some practical examples in which components and systems have deteriorating states for which we cannot say one deteriorating state is better/worse than other one. Then a model of reliability systems assuming partially ordered sets for the state spaces is useful,

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and Yu, Koren and Guo [25] presented a model of such cases and gave some properties of them, but not fully.

A mathematical generalization to partially ordered set cases, aiming at giving a theoretical framework of multi-state systems, is given by F.Ohi [19], giving existence theorem of series and parallel systems and decomposition of systems structure by series systems, which is well known for the binary case as "max − min" formulae with minimal path or cut sets, see Barlow and Proschan [1].

In this work, a continuation of our recent work F.Ohi [19], using associated probability measures on partially ordered sets, see F.Ohi, S.Shinmori and T.Nishida [16], and series decomposition of multi-state systems, we give stochastic bounds for reliability performances of multi-state systems.

The following sections 2 and 3 give a summary of the results of F.Ohi [19] which are needed for giving stochastic bounds in section4.

2. Coherent Systems

Definition 1. (Definition of a System) A system composed of n components (a system of order n) is a triplet (Ω_C, S, φ) satisfying the following conditions.

(1) $C = \{1, \cdots, n\}$ is the set of integers from 1 to n, where each number is the index of each unit.

(2) Ω_i $(i \in C)$ is a finite lattice set having the least and the greatest elements denoted by m_i and M_i, respectively.

(3) $\Omega_C = \prod_{i=1}^{n} \Omega_i$ is the product lattice set of Ω_i $(i \in C)$. Each element $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega_C$, which is called a state vector, means a combination of states of the components as $x_i \in \Omega_i$ is the state of the $i-$th component.

(4) S is a finite lattice set having the least and the greatest elements denoted by m and M, respectively.

(5) φ is a surjection from Ω_C to S, which is also called a structure function. For a state vector $\boldsymbol{x} \in \Omega_C$, $\varphi(\boldsymbol{x})$ is the state of the system.

The least and the greatest elements mean the full failure and the perfect functioning states, respectively.

In the sequel we sometimes use a notation for a state vector as

$$(k_i, \boldsymbol{x}) = (x_1, \cdots, x_{i-1}, k, x_{i+1}, \cdots, x_n), \quad k \in \Omega_i$$

to highlight the state of the $i-$th component as k.

The orders on Ω_i $(i \in C), S$ are denoted by the common symbol $\leqq$, and $a < b$ is used for a and b of Ω_i $(i \in C)$ or S to mean $a \leqq b$ and $a \neq b$. The symbols $\wedge$ and $\vee$ denote inf and sup, respectively.

For $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$ of Ω_C, $\boldsymbol{x} \leqq \boldsymbol{y}$ means $x_i \leqq y_i$ $(\forall i \in C)$ and $\boldsymbol{x} < \boldsymbol{y}$ means $x_i < y_i$ $(\forall i \in C)$. We notice for $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C, $\boldsymbol{x} \lneqq \boldsymbol{y}$ means that $x_i \leqq y_i$ $(\forall i \in C)$ and $x_j < y_j$ for some $j \in C$, and is not necessarily same to $\boldsymbol{x} < \boldsymbol{y}$. On the other hand for x and y of Ω_i $(i \in C)$ or S, $x \lneqq y$ is equivalent to $x < y$. Furthermore since $\Omega_i (i \in C)$ and S are assumed to be lattice sets, then we may define the infimum and the maximum of two state vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\boldsymbol{x} \wedge \boldsymbol{y} \stackrel{def}{=} (x_1 \wedge y_1, \cdots, x_n \wedge y_n), \quad \boldsymbol{x} \vee \boldsymbol{y} \stackrel{def}{=} (x_1 \vee y_1, \cdots, x_n \vee y_n),$$

where $\wedge$ and $\vee$ mean the infimum and the maximum, respectively.

A system (Ω_C, S, φ) is simply called a system φ when there is no confusion. For a system φ and an element $s \in S$, we define

$$\begin{aligned} V_{s\leqq}(\varphi) &\stackrel{def}{=} \{\boldsymbol{x} \in \Omega_C \mid \varphi(\boldsymbol{x}) \geqq s\}, \\ V_{\leqq s}(\varphi) &\stackrel{def}{=} \{\boldsymbol{x} \in \Omega_C \mid \varphi(\boldsymbol{x}) \leqq s\}, \\ V_s(\varphi) &\stackrel{def}{=} \{\boldsymbol{x} \in \Omega_C \mid \varphi(\boldsymbol{x}) = s\}. \end{aligned}$$

Sometimes omitting φ, we simply write, for example, $V_{s\leqq}(\varphi)$ as $V_{s\leqq}$.

For a while, we assume A to be an ordered set. Generally $MI(A)$ and $MA(A)$ are the sets of all the minimal and maximal elements of A, respectively. For $a, b \in A$, b is called a successor of a if $a < b$ and there is no element c of A such that $a \leqq c \leqq b$, $a \neq c$ and $b \neq c$, in which case a is also called a predecessor of b. A successor b of a is sometimes written as $a+1$ and a as $b-1$. We notice that neither a predecessor nor a successor is uniquely determined in general, since the state spaces are assumed to be lattice sets and then not necessarily to be totally ordered sets. $S(a)$ is defined to be the set of all the successors of a.

For r and t of A such that $r < t$, a path of length k from r to t is a series $(s_0, s_1, \cdots, s_k)$ satisfying $s_0 = s < s_1 < \cdots < s_k = t$ and $s_{i+1} = s_i + 1$, $i = 0, 1, \cdots, k$. $d(A)$ is used to denote the longest length among the paths from minimal to maximal elements of A.

Definition 2. (Increasing Property) A system φ is called increasing, when for $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C, $\boldsymbol{x} \leqq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \leqq \varphi(\boldsymbol{y})$.

Definition 3. (Normal Property) A system φ is called normal when for every $s \in S$,

$$\forall \boldsymbol{x} \in MI(V_{s\leqq}),\ \varphi(\boldsymbol{x}) = s, \tag{1}$$

$$\forall \boldsymbol{x} \in MA(V_{\leqq s}),\ \varphi(\boldsymbol{x}) = s, \tag{2}$$

in other words (3) is equivalent to $MI(V_{s\leqq}) = MI(V_s)$ and (4) to $MA(V_{\leqq s}) = MA(V_s)$.

Definition 4. (Relevant Property) (1) The component $i \in C$ is said to be relevant to the system when the following is satisfied.

$$\forall r \in S, \forall s \in S(r), \exists (k_i, \boldsymbol{x}), \exists (l_i, \boldsymbol{x}) \text{ such that } k < l, \quad \varphi(k_i, \boldsymbol{x}) = r, \varphi(l_i, \boldsymbol{x}) = s.$$

(2) A system φ is called relevant when every component is relevant to the system.

Definition 5. (Coherent System) A system φ is called coherent, when φ is increasing, normal and relevant.

For an increasing system φ, the condition in (1) of Definition 4 is equivalent to

$$\forall r \in S, \forall s \in S(r),\ \exists \boldsymbol{x} \in \Omega_C,\ \varphi(\boldsymbol{x}) = s,\ \varphi(x_i - 1, \boldsymbol{x}) = r.$$

Definition 6. (Weakly Relevant Property) (1) The component $i \in C$ is said to be weakly relevant to the system when the following is satisfied.

$$\exists r \in S, \exists s \in S(r), \exists (k_i, \boldsymbol{x}), \exists (l_i, \boldsymbol{x}) \text{ such that } k < l, \quad \varphi(k_i, \boldsymbol{x}) = r, \varphi(l_i, \boldsymbol{x}) = s.$$

(2) A system φ is called weakly relevant when every component is weakly relevant.

The condition of Definition 6 is apparently weaker than that of Definition 5 and has practically no restriction on the systems, since if the condition of Definition 6 does not hold for a component $i \in C$, then we have

$$\forall \boldsymbol{x} \in \Omega_{C\setminus\{i\}}, \forall k, \forall l \in \Omega_i,\ \varphi(k_i, \boldsymbol{x}) = \varphi(l_i, \boldsymbol{x}),$$

which means that the states of the component i does not contribute to the performance of the system at all and then we can delete the component i from the system. **In the sequel we assume φ to be weakly relevant without generality.**

3. Series Systems and Decomposition of Systems

In this section we present a definition of series and parallel systems for the case of finite lattice, not necessarily totally ordered sets, and a decomposition of systems by series systems is given which is used in the next section to give stochastic bounds for reliability performances of systems.

Noticing that Ω_i $(i \in C)$ and S are finite lattice sets, we have infimum and supremum elements for every subset of Ω_i $(i \in C), S$ and Ω_C.

Definition 7. (Series and Parallel Systems) Let φ be an increasing system.

(1) φ is called a series system if $V_{s\leqq}$ has the least element which is simply written as $\boldsymbol{m}_s$.

(2) φ is called a parallel system if $V_{\leqq s}$ has the greatest element which is simply written as $\boldsymbol{M}_s$.

Since the parallel systems are the dual of series systems, in the sequel we focus on the series systems.

When φ is a series system, $\varphi(\min V_{s\leqq}) = s$ for every $s \in S$, since the system φ is surjective, then the system φ is necessarily normal.

In the following we assume (Ω_C, S, φ) to be an increasing, normal and weakly relevant system. Letting $\boldsymbol{p} = (s_0, s_1, \cdots, s_k)$ be a path from m to M of S, so $s_0 = m$ and $s_k = M$, we define $\varphi_{\boldsymbol{p}} : \Omega_C \to S_{\boldsymbol{p}} = \{s_0, s_1, \cdots, s_k\}$ as

$$\boldsymbol{x} \in \Omega_C,\ \varphi_{\boldsymbol{p}}(\boldsymbol{x}) = \max\{\ s_l \mid s_l \leq \varphi(\boldsymbol{x})\ \}\ .$$

The parenthesis is not empty, since $\varphi(\boldsymbol{x}) \geq s_0 = m$.

Then we have $\varphi_{\boldsymbol{p}}$ clearly satisfying

$$\varphi(\boldsymbol{x}) = s \implies \varphi_{\boldsymbol{p}}(\boldsymbol{x}) \begin{cases} = s & \text{if} \quad s \in S_{\boldsymbol{p}}, \\ \leqq s & \text{if} \quad s \notin S_{\boldsymbol{p}}\ . \end{cases} \tag{3}$$

Letting $\mathcal{P}$ be the set of all the paths from m to M on S and noticing that for every $s \in S$ there exists a path from m to M which contains s, from (3) we have for every $\boldsymbol{x} \in \Omega_C$,

$$\varphi(\boldsymbol{x}) = \max_{\boldsymbol{p}\in\mathcal{P}} \varphi_{\boldsymbol{p}}(\boldsymbol{x})\ . \tag{4}$$

Furthermore we decompose $\varphi_{\boldsymbol{p}}$. For a path $\boldsymbol{p} = (s_0, s_1, \cdots, s_k)$ from m to M on S, let $\mathcal{K}_{\boldsymbol{p}}$ be the set of all maximal paths of $\cup_{s\in S_{\boldsymbol{p}}} MIV_s$. Notice that the initial element of each path of $\mathcal{K}_{\boldsymbol{p}}$ is $\boldsymbol{m} = (m_1, \cdots, m_n)$. For a sequence $\boldsymbol{k} = (\boldsymbol{m}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_k) \in \mathcal{K}_{\boldsymbol{p}}$, we define a series system $\varphi^{\boldsymbol{p}}_{\boldsymbol{k}} : \Omega_C \to S_{\boldsymbol{p}} = \{s_0, s_1, \cdots, s_k\}$ as

$$\varphi^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}) \stackrel{def}{=} s, \quad \text{where } \max\{\ \boldsymbol{x}_l \mid \boldsymbol{x}_l \leq \boldsymbol{x}\ \} \in MIV_s.$$

Then we have the following formula.

$$\varphi_{\boldsymbol{p}}(\boldsymbol{x}) = \max_{\boldsymbol{k}\in\mathcal{K}_{\boldsymbol{p}}} \varphi^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x})\ . \tag{5}$$

Combining (4) and (5), we finally have a decomposition of the system φ by series systems $\varphi^{\boldsymbol{p}}_{\boldsymbol{k}}$, $\boldsymbol{p} \in \mathcal{P}, \boldsymbol{k} \in \mathcal{K}_{\boldsymbol{p}}$.

Theorem 1 *For every $\boldsymbol{x} \in \Omega_C$,*

$$\varphi(\boldsymbol{x}) = \max_{\boldsymbol{p}\in\mathcal{P}} \max_{\boldsymbol{k}\in\mathcal{K}_{\boldsymbol{p}}} \varphi^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}).$$

4. Stochastic Bounds for Systems

In this section we give stochastic bounds for stochastic performances of a multi-state system $(\prod_{i=1}^n \Omega_i, S, \varphi)$, which is a generalization of those well known for the binary-state coherent systems expressed by using minimal path series systems.

First we give a general definition of associated probability on an ordered set before showing stochastic bounds for multi-state systems.

Definition 8. (Ohi, Shinmori and Nishida [16]) A probability $\boldsymbol{Q}$ on an ordered set Ω is called associated when the following condition is hold.

$$\forall \text{ increasing subsets } A \text{ and } B \text{ of } \Omega, \quad \boldsymbol{Q}(A \cap B) \geq \boldsymbol{Q}(A)\boldsymbol{Q}(B)\ .$$

Noticing that $A \subset \Omega$ is an increasing set if and only if $A^c = \Omega \backslash A$ is a decreasing set, we have an equivalent definition of an associated probability as the following.

$$\forall \text{ decreasing subsets } A \text{ and } B \text{ of } \Omega, \quad Q(A \cap B) \geq Q(A)Q(B) .$$

Lemma 1 *For an associated probability Q on an ordered set Ω,*

$$\forall \textit{ increasing subsets } A \textit{ and } B \textit{ of } \Omega, \quad Q(A \cup B) \leq 1 - (1 - Q(A))(1 - Q(B)) .$$

PROOF. Since A^c, B^c are decreasing sets for increasing sets A and B, we have

$$Q(A \cup B) = 1 - Q(A^c \cap B^c) \leq 1 - Q(A^c)Q(B^c) = 1 - (1 - Q(A))(1 - Q(B)) .$$

Theorem 2 *Let P be an associated probability on $\prod_{i=1}^{n} \Omega_i$, where the state spaces are finite sets, then the power set is taken to be the σ-field . Using the decomposition of systems by series systems, we have the following inequalities.*

$$\prod_{p \in \mathcal{P}} \prod_{k \in \mathcal{K}_p} P\left(\varphi_k^p \leq s\right) \leq P(\varphi \leq s) \leq \min_{p \in \mathcal{P}} \min_{k \in \mathcal{K}_p} P\left(\varphi_k^p \leq s\right),$$

$$\max_{p \in \mathcal{P}} \max_{k \in \mathcal{K}_p} P\left(\varphi_k^p \geq s\right) \leq P(\varphi \geq s) \leq 1 - \prod_{p \in \mathcal{P}} \{1 - P(\varphi^p \geq s)\}$$
$$\leq 1 - \prod_{p \in \mathcal{P}} \prod_{k \in \mathcal{K}_p} \left\{1 - P\left(\varphi_k^p \geq s\right)\right\}.$$

PROOF. Noticing that φ_k^p, φ^p $(p \in \mathcal{P},\ k \in \mathcal{K}_p)$ are increasing and P is an associated probability measure, the proof is easy.

$$P(\varphi \leq s) = P\left(\max_{p \in \mathcal{P}} \max_{k \in \mathcal{K}_p} \varphi_k^p \leq s\right) = P\left(\bigcap_{p \in \mathcal{P}} \bigcap_{k \in \mathcal{K}_p} \left\{\varphi_k^p \leq s\right\}\right) \geq \prod_{p \in \mathcal{P}} \prod_{k \in \mathcal{K}_p} P\left(\varphi_k^p \leq s\right) .$$

Since the following relationships hold,

$$\varphi(x) \leq s \iff \max_{p \in \mathcal{P}} \max_{k \in \mathcal{K}_p} \varphi_k^p(x) \leq s \iff \forall p \in \mathcal{P},\ \forall k \in \mathcal{K}_p,\ \varphi_k^p(x) \leq s$$

and then $\forall p \in \mathcal{P}$, $\forall k \in \mathcal{K}_p$

$$\varphi(x) \leq s \implies \varphi_k^p(x) \leq s .$$

Hence we have the following inequality:

$$P(\varphi \leq s) \leq \min_{p \in \mathcal{P}} \min_{k \in \mathcal{K}} P\left(\varphi_k^p \leq s\right).$$

The second and the third inequalities of the second inequality-chain are easily obtained by Lemma 1. The first inequality of the chain is given by noticing the following relations.

Since $\varphi(x) = \max_{p \in \mathcal{P}} \max_{k \in \mathcal{K}_p} \varphi_k^p(x)$, then we have

$$\forall p \in \mathcal{P},\ \forall k \in \mathcal{K}_p,\ \varphi_k^p(x) \leq \varphi(x)$$

and

$$\forall p \in \mathcal{P},\ \forall k \in \mathcal{K}_p,\ P\left(\varphi_k^p \geq s\right) \leq P(\varphi \geq s).$$

Thus the first inequality of the second chain holds.

Corollary 1 *When the probability P on $\prod_{i=1}^{n} \Omega_i$ is given as the product probability of associated probabilities P_i on Ω_i $(i = 1, \cdots, n)$, then P is associated and the similar probability bounds are given for the system.*

5. Concluding Remarks

In this paper we have given stochastic bounds for reliability performances of multi-state systems, using the decomposition of system structure functions by series systems (F.Ohi [19]) and associated probability measures on partially ordered sets (F.Ohi, S Shinmori and T.Nishida [16]). Numerical examinations and evaluating degree of the approximation of the given bounds are remained to be a future problem.

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